Introduction to MHD

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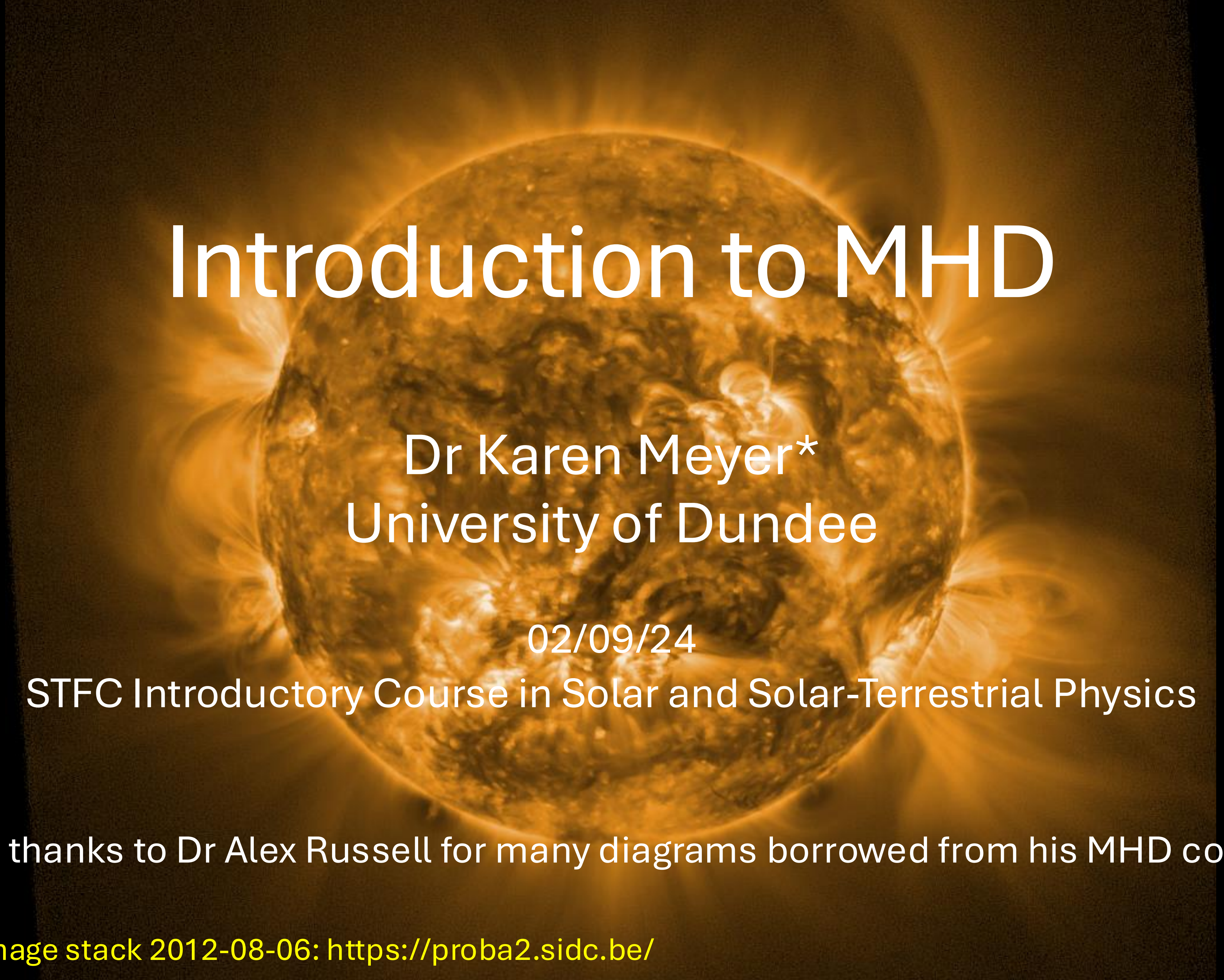
STFC Introductory Course in Solar and Solar-Terrestrial Physics

*with thanks to Dr Alex Russell for many diagrams borrowed from his MHD course!



PROBA2/SWAP image stack 2012-08-06: https://proba2.sidc.be/

Magnetohydrodynamics (MHD)

magneto-

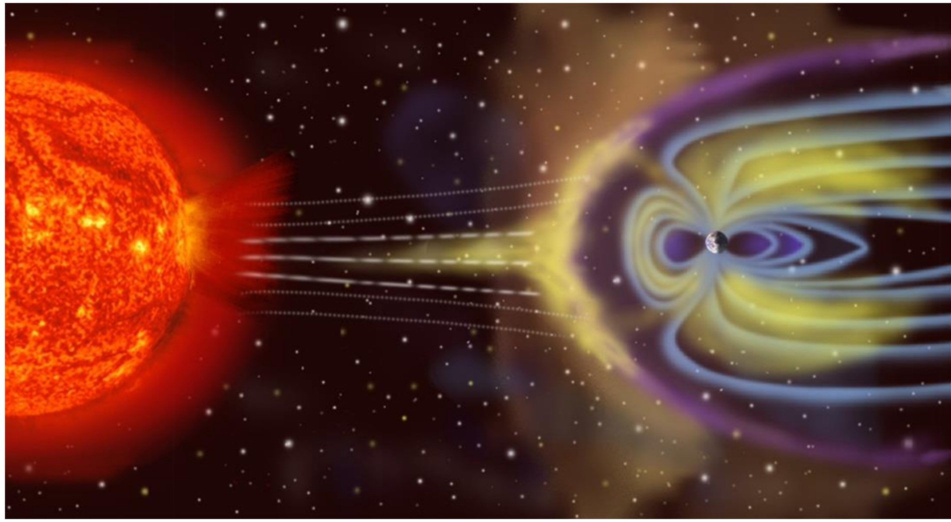
-hydrodynamics

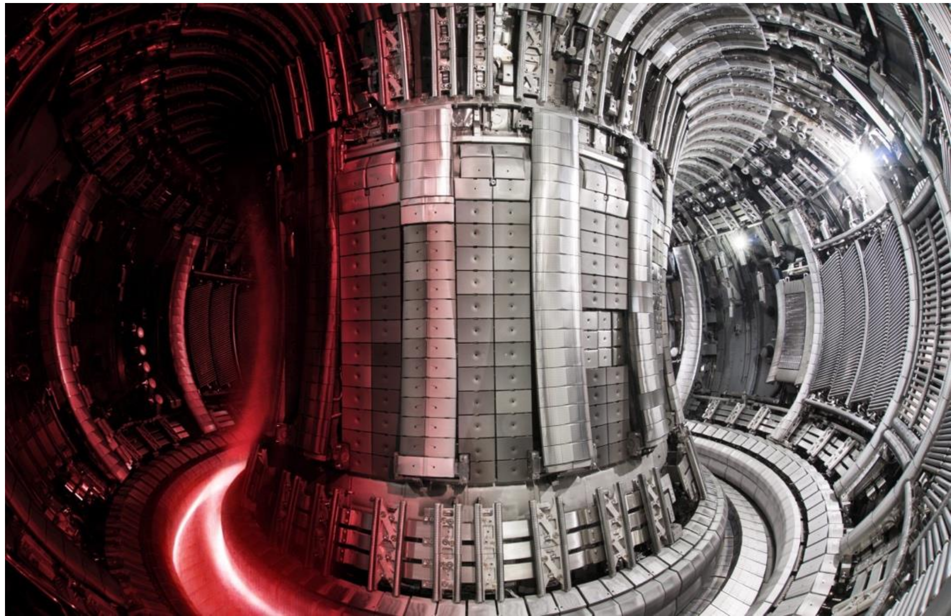
≡ electromagnetism

≡ fluid dynamics

Magnetohydrodynamics (MHD)

- Couples Maxwell's equations of electromagnetism with hydrodynamics
- Describes macroscopic behaviour of conducting fluids such as plasmas
- Important e.g. for solar physics, magnetospheric physics, astrophysics, laboratory plasma experiments





Top: Sun-Earth connection (www.esa.int)
Bottom: Interior of JET tokamak in Culham, UK, with superimposed image of hot plasma (ccfe.ukaea.uk)

Fluid Equations

Continuity equation (mass conservation): classically, matter is neither created nor destroyed

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

ρ = fluid mass density
$\mathbf{v}$ = fluid velocity

Fluid Equations

Equation of motion (momentum conservation, Newton's 2nd Law):

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \rho \mathbf{g},$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

is the total time derivative or convective derivative.

p = pressure
$\mathbf{g}$ = gravitational acceleration

Fluid Equations

Fluid energy equation (energy conservation):

$$\frac{Dp}{Dt} = -\gamma p \nabla \cdot \mathbf{v} + (\gamma - 1)Q$$

γ = ratio of specific heats, normally $\gamma = 5/3$
Q = heating rate, can contain positive (heating), negative (cooling e.g. by radiation) and "movement" (e.g. thermal conduction) terms

$Q = 0$: adiabatic case

Summary of Fluid Equations

Mass Continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Equation of motion:

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \rho \mathbf{g},$$

Energy equation:

$$\frac{Dp}{Dt} = -\gamma p \nabla \cdot \mathbf{v} + (\gamma - 1)Q$$

Maxwell's Equations

	$\mathbf{B}$	$\mathbf{E}$
Div	Solenoidal Constraint $\nabla \cdot \mathbf{B} = 0$	Gauss' Law $\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0}$
Curl	Ampère's Law $\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}$	Faraday's Law $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

$\mathbf{B}$ = magnetic field
$\mathbf{E}$ = electric field
$\mathbf{j}$ = electric current density
ρ_c = charge density
ϵ_0 = permittivity of free space
c = speed of light in vacuum

Coupling the Fluid and EM equations

Ohm's Law (for moving conductors):

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

σ = conductivity of material

- Conducting fluid can change electromagnetic fields
- How do electromagnetic fields affect the fluid?

Electromagnetic energy and Poynting's Theorem (1984)

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{j}, \qquad U = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}, \qquad \boldsymbol{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

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How is energy converted with Ohm's law?

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad \Leftrightarrow \quad \boldsymbol{E} = -\mathbf{v} \times \mathbf{B} + \frac{\boldsymbol{j}}{\sigma}$$

Resistive heating

$$-\mathbf{E} \cdot \mathbf{j} = \mathbf{v} \times \mathbf{B} \cdot \mathbf{j} - \frac{\mathrm{j}^2}{\sigma} = \mathbf{v} \cdot \mathbf{B} \times \mathbf{j} - \frac{\mathrm{j}^2}{\sigma} = -\mathbf{v} \cdot \mathbf{j} \times \mathbf{B} - \frac{j^2}{\sigma}$$

Work done on conductor

Electromagnetic forces

Lorentz force on a charged particle (microscopic):

$$\mathbf{F}_{\mathrm{L}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Lorentz force density over fluid element (macroscopic):

$$\boldsymbol{f}_L = \frac{1}{\delta V}\sum_i \boldsymbol{F}_{L\,i} = \left(\frac{1}{\delta V}\sum_i q_i\right)\mathbf{E} + \left(\frac{1}{\delta V}\sum_i q_i\,\mathbf{v}_i\right) \times \mathbf{B} = \rho_c\mathbf{E} + \mathbf{j} \times \mathbf{B}$$

Summary of key EM results

Maxwell's equations:

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0}, \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

Ohm's law for conductors:

$$\boldsymbol{E} = -\mathbf{v} \times \mathbf{B} + \frac{\boldsymbol{j}}{\sigma}$$

Lorentz force:

$$\boldsymbol{f}_L = \rho_c \mathbf{E} + \mathbf{j} \times \mathbf{B}$$

Poynting Theorem:

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{j}, \qquad U = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0},$$

$$\boldsymbol{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

Heating:

$$Q_{\mathrm{ohmic}} = \frac{j^2}{\sigma}$$

MHD Assumptions

MHD considers phenomena with typical speeds (e.g. flow speeds, wave speeds) much less than the speed of light:

$$v_0 \ll c$$

⇒ simplifications to Maxwell's equations

[Note: we are also assuming a fully ionised hydrogen plasma]

Order of Magnitude Analysis

Assume B_0, E_0, l_0, t_0 to be typical values for magnetic and electric field strength, length scale and time scale.

$$\Rightarrow \qquad v_0 = \frac{l_0}{t_0} \quad \text{typical speed } (\ll c)$$

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Comparing magnitude of terms in Faraday's law:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \quad \frac{E_0}{l_0} \sim \frac{B_0}{t_0} \quad \Rightarrow \quad E_0 \sim v_0 B_0$$

Order of Magnitude Analysis

Now, comparing terms on LHS of Ampère's law:

$E_0 \sim v_0 B_0$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}: \qquad \frac{\left|\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right|}{|\nabla \times \mathbf{B}|} \sim \frac{E_0/(t_0 c^2)}{B_0/l_0} \sim \left(\frac{v_0}{c}\right)^2 \ll 1$$

Order of Magnitude Analysis

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⇒ Neglect displacement current term.

MHD Ampère's law:

$$\mathbf{j} = \frac{1}{\mu_0}\nabla \times \mathbf{B}$$

Order of Magnitude Analysis

It can similarly be shown that $|\rho_c \mathbf{E}| \ll |\mathbf{j} \times \mathbf{B}|$

⇒ Lorentz force density $\boldsymbol{f}_L = \rho_c \mathbf{E} + \mathbf{j} \times \mathbf{B}$ reduces to

MHD Lorentz force:

$$\boldsymbol{f}_L = \mathbf{j} \times \mathbf{B}$$

Remarks

1. Simplifying Lorentz force density $\Rightarrow$ we are effectively assuming $\rho_c \approx 0$, which is called quasi-neutrality.

2. Assumption $v_0 \ll c$ excludes electromagnetic waves (can still get acoustic and magnetic waves – see talk by Chris Nelson at 2pm).

3. Since MHD uses a fluid description, l_0 and t_0 need to be large enough that local averages become meaningful:
   i. $l_0 \gg$ gyroradii of particles
   ii. $t_0 \gg$ gyroperiods of particles
   iii. Plasma density ρ must be sufficiently high.

Bringing it all together

Fluid Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \rho \mathbf{g},$$

$$\frac{Dp}{Dt} = -\gamma p \nabla \cdot \mathbf{v} + (\gamma - 1) Q$$

EM equations

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0}, \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\boldsymbol{f}_L = \rho_c \mathbf{E} + \mathbf{j} \times \mathbf{B}$$

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Bringing it all together

Fluid Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

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EM equations

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$\left(\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0}, \right) \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\boldsymbol{E} = -\mathbf{v} \times \mathbf{B} + \frac{\boldsymbol{j}}{\sigma}$$

Resistive MHD Equations

Mass continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Momentum equation:

$$\rho \frac{D\mathbf{v}}{Dt} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p + \rho \mathbf{g},$$

Energy equation:

$$\frac{Dp}{Dt} = -\gamma p \nabla \cdot \mathbf{v} + (\gamma - 1)\eta \frac{|\nabla \times \mathbf{B}|^2}{\mu_0}$$

Induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

4 PDEs in 4 primary variables: $\mathbf{B}, \mathbf{v}, \rho$, p

Obtain secondary variables $\mathbf{E}$ and $\mathbf{j}$ from MHD Ampère and Ohm's laws.

$$\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B}$$

Also:

$$\nabla \cdot \mathbf{B} = 0$$

$$\eta = \frac{1}{\mu_0 \sigma}$$

Magnetic diffusivity or resistivity

Induction Equation and R_m

Induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \underbrace{\nabla \times (\mathbf{v} \times \mathbf{B})}_{\text{advection}} + \underbrace{\eta \nabla^2 \mathbf{B}}_{\text{diffusion}}$$

Comparing magnitude of terms:

$$\frac{|\nabla \times (\mathbf{v} \times \mathbf{B})|}{|\eta \nabla^2 \mathbf{B}\,|} \sim \frac{v_0 B_0 / l_0}{\eta B_0 / l_0^2} \sim \frac{l_0 v_0}{\eta} =: R_m$$

R_m is the magnetic Reynolds number.

Large values common in solar corona (e.g. $10^8 - 10^{12}$).

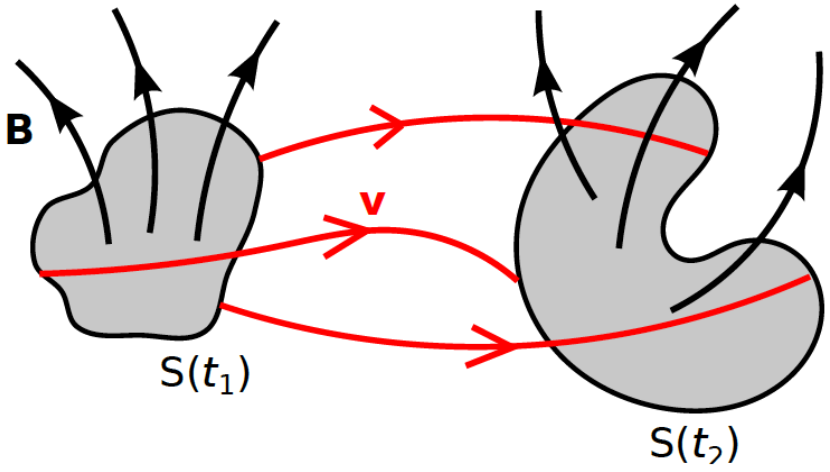
Ideal MHD

When $R_m \gg 1$ use ideal induction equation:

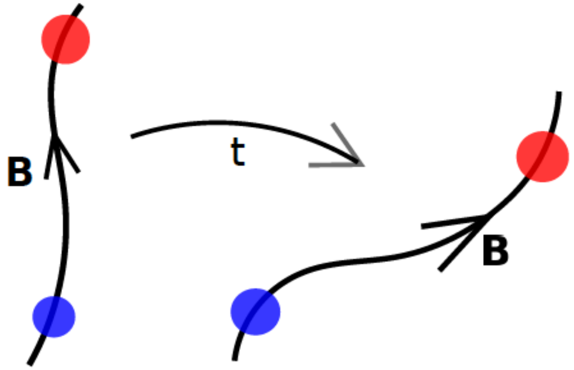
$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

Alfvén's Theorem:
The magnetic flux through any surface co-moving with $\mathbf{v}$ is conserved.

$$\frac{d}{dt}\int_{S(t)} \mathbf{B} \cdot dS = 0$$



Field line conservation:
Two fluid elements lying on the same magnetic field line always do so.



Magnetic Forces

MHD momentum equation:

$$\rho\left(\frac{d\mathbf{v}}{dt} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = \mathbf{j}\times\mathbf{B} - \nabla p + \rho\mathbf{g}$$

Magnetic Forces

MHD momentum equation:

$$\rho\left(\frac{d\mathbf{v}}{dt}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right)=\boxed{\mathbf{j}\times\mathbf{B}}-\nabla p+\rho\mathbf{g}$$

Lorentz force term can be expanded as

$$\mathbf{j}\times\mathbf{B}=\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}=\underbrace{-\nabla\left(\frac{B^2}{2\mu_0}\right)}_{\text{Magnetic pressure}}+\underbrace{\frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B}}_{\text{Magnetic tension}}$$

Magnetic Forces

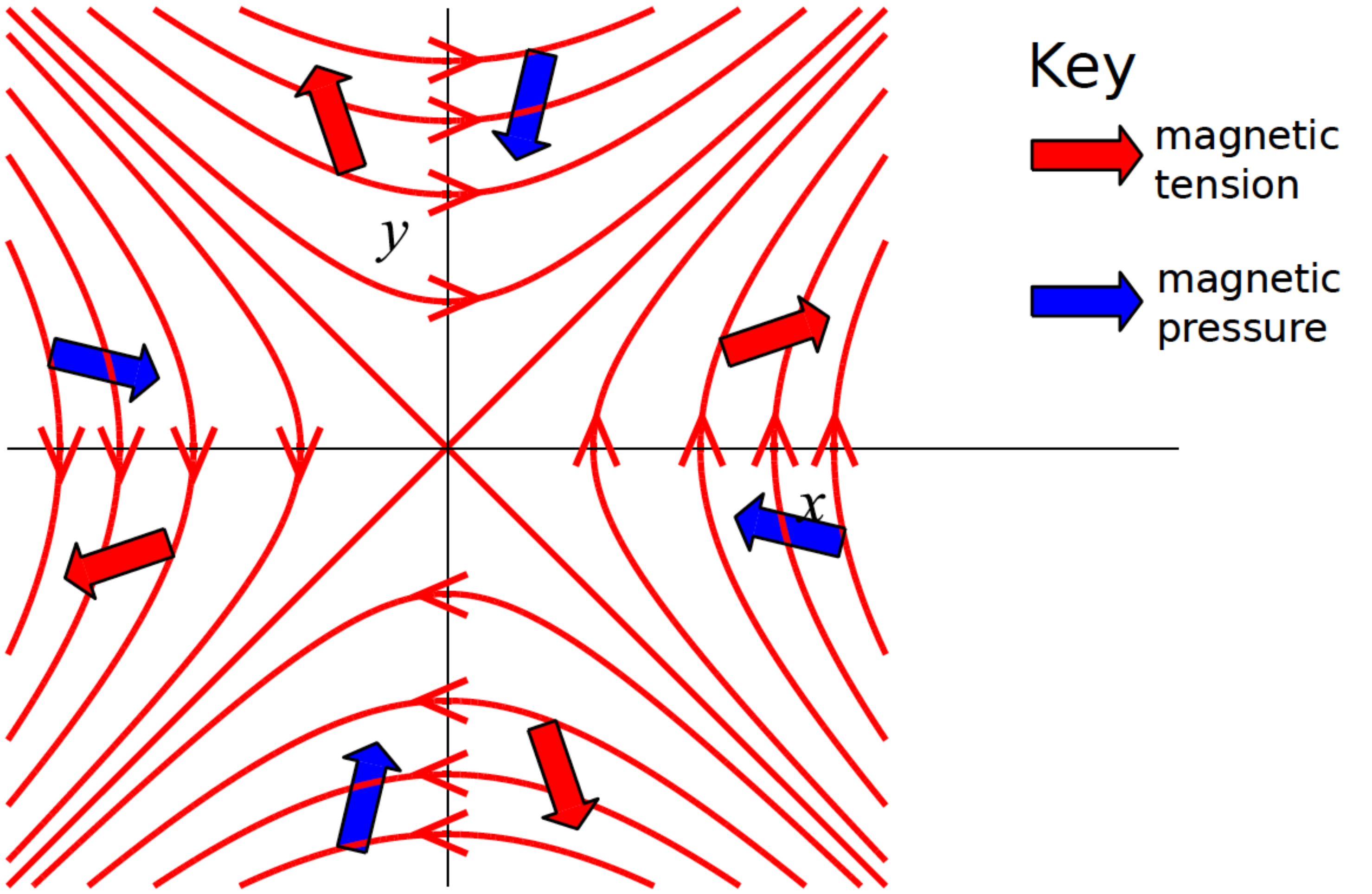
E.g. $\mathbf{B} = B_0\left(y\boldsymbol{e}_x + x\boldsymbol{e}_y\right)$

Magnetic pressure:

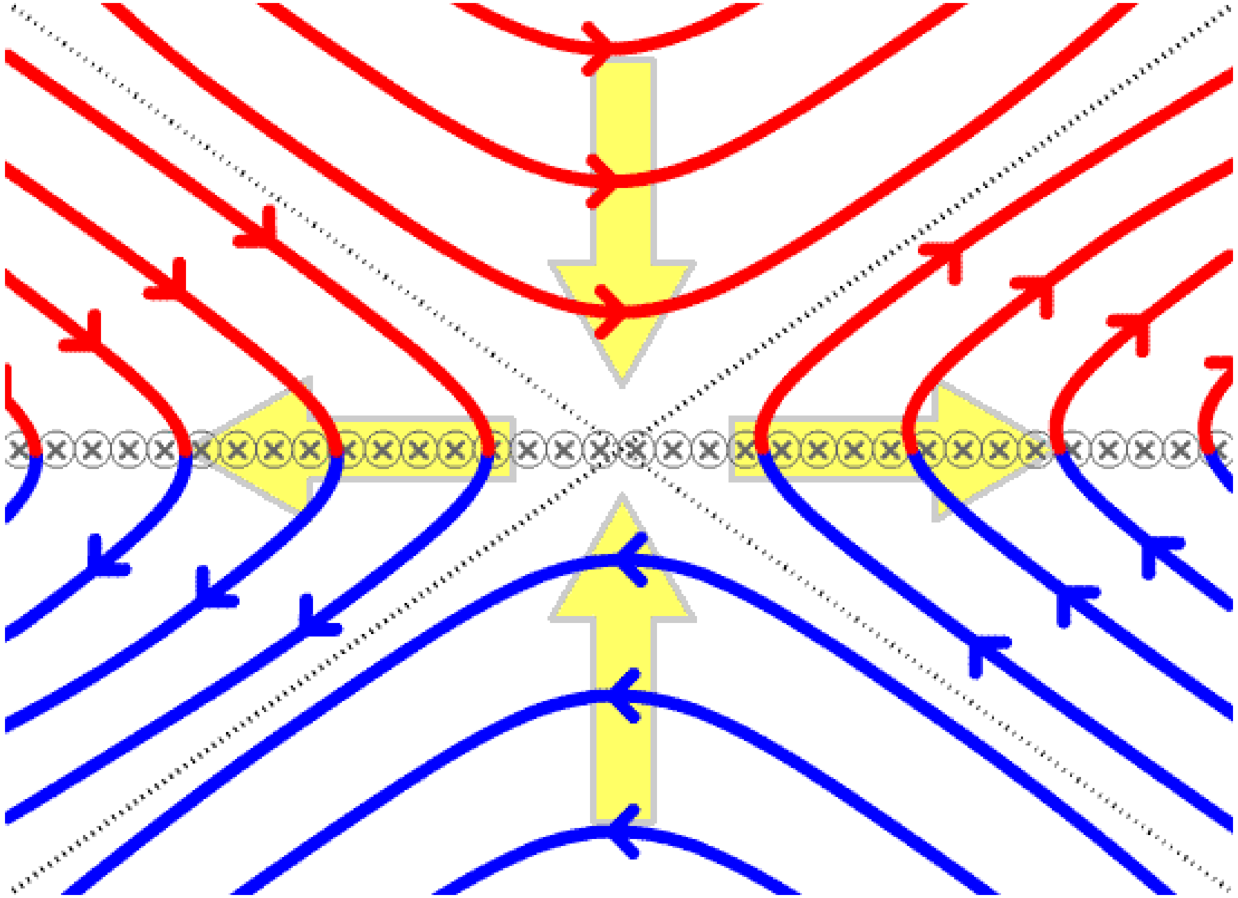
$$-\nabla\left(\frac{B^2}{2\mu_0}\right) = -\frac{B_0^2}{\mu_0}\boldsymbol{e}_r$$

Magnetic tension:

$$\frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} = \frac{B_0^2}{\mu_0}\boldsymbol{e}_r$$



Magnetic Reconnection



See Peter Wyper's talk at 4pm

en.wikipedia.org/wiki/Magnetic_reconnection

Force-Free Fields

MHD momentum equation:

$$\rho\left(\frac{d\mathbf{v}}{dt} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = \mathbf{j}\times\mathbf{B} - \nabla p + \rho\mathbf{g}$$

Force-Free Fields

MHD momentum equation:

Consider long-lived structures: ~static equilibrium

$$\rho\left(\frac{d\mathbf{v}}{dt} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = \mathbf{j}\times\mathbf{B} - \nabla p + \rho\mathbf{g}$$

Effect of gravity often negligible compared to other forces

Plasma beta is pressure ratio: $\beta := \dfrac{p}{B^2/2\mu_0}$

$\beta \ll 1$ when plasma pressure $\ll$ magnetic pressure.

Seek static equilibria where

$$\mathbf{j}\times\mathbf{B} = \mathbf{0}$$

$\Rightarrow$ Force balance between magnetic pressure and magnetic tension.

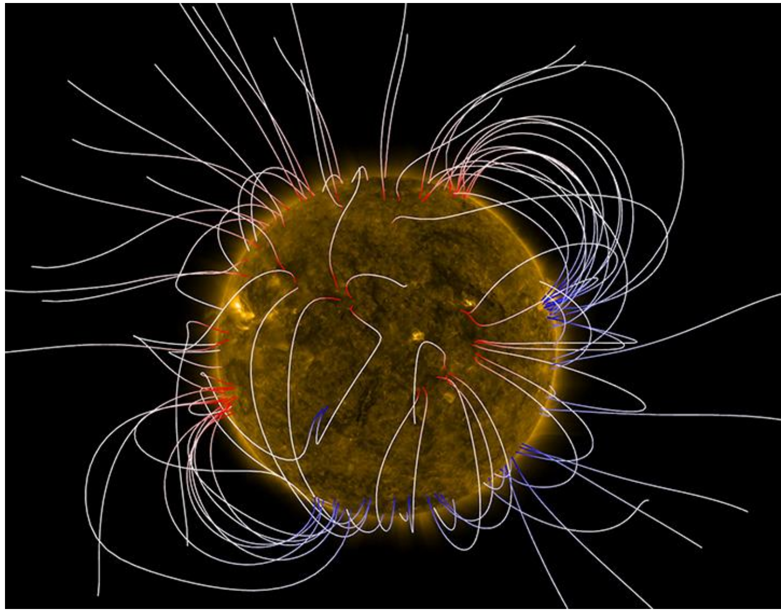
Force-Free Fields

$$\mathbf{j} \times \mathbf{B} = \mathbf{0} \quad \Leftrightarrow \quad (\nabla \times \mathbf{B}) \times \mathbf{B} = \mathbf{0} \quad \Leftrightarrow \quad \nabla \times \mathbf{B} = \alpha(\mathbf{x})\mathbf{B}$$

Taking divergence:

$$\mathbf{B} \cdot \nabla\alpha = 0$$

i.e. α is constant along magnetic field lines.



sdo.gsfc.nasa.gov

Cases:

$\alpha = 0$:	potential field
$\alpha = \text{const everywhere}$:	linear force-free field
$\alpha = \alpha(\mathbf{x})$:	non-linear force-free field

Extensions

Many possible extensions, keeping fluid approximation:

- Generalised Ohm's law: short lengthscales, terms related to electrons and ions.
- Different electron and ion temps.
- Anisotropic thermal conduction.
- Two-fluid plasma, e.g. including neutrals.
- Relativistic MHD
- …

Kinetic theory: abandon fluid approximation and track particle distribution function – or a hybrid approach.

Summary

Mass continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Momentum equation:

$$\rho \frac{D\mathbf{v}}{Dt} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p + \rho \mathbf{g},$$

Energy equation:

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Also:

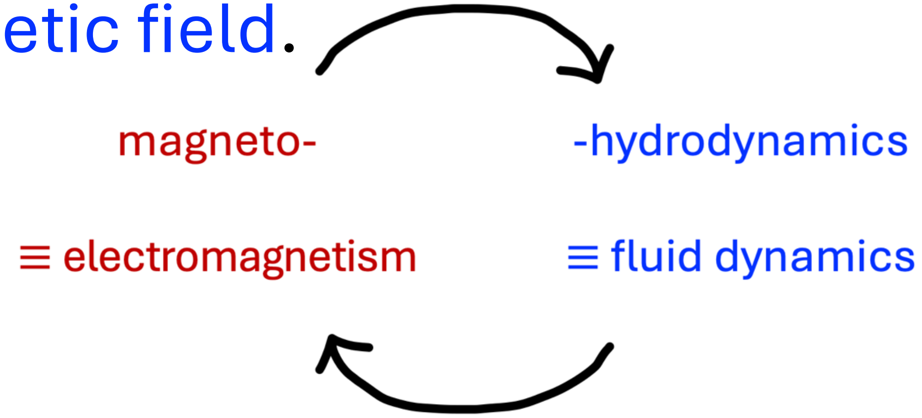
$$\nabla \cdot \mathbf{B} = 0$$

$$\eta = \frac{1}{\mu_0 \sigma}$$

Magnetic diffusivity or resistivity

Summary

- Magnetohydrodynamics (MHD) describes interaction of an electrically conducting fluid with a magnetic field.



- Assumptions (resistive MHD):
  - Fully ionized (single fluid) plasma
  - $v_0 \ll c$
  - $\rho_c \approx 0$ (quasi-neutrality)
  - "Large enough" length and time scales ($l_0 \gg$ gyroradii, $t_0 \gg$ gyroperiods of particles, density ρ sufficiently high).

- Ideal MHD when $R_m := l_0 v_0/\eta \gg 1$: field lines "frozen-in".
- Force-free assumption valid when $\beta := p/(B^2/2\mu_0) \ll 1$.

Thanks for listening! Any questions?

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Dundee from Dundee Law